A Probability Primer for Robotics Students

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Tutorial Goal

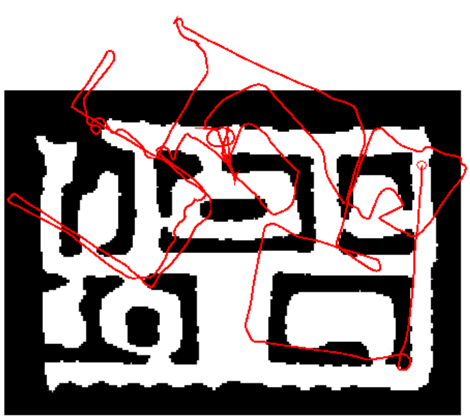
To familiarize you with probabilistic paradigm in robotics

- Axioms
- Basic techniques
- Bayes rule
- Recursive Bayesian updating
- Bayes filters

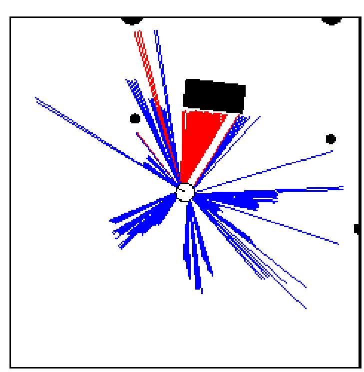
Robots are Inherently Uncertain

- Uncertainty arises from four major factors:
  - Environment stochastic, unpredictable
  - Robot stochastic
  - Sensor limited, noisy
  - Models inaccurate

Nature of Sensor Data



Odometry Data



Range Data

Advantages of Probabilistic Paradigm

- Can accommodate inaccurate models
- Can accommodate imperfect sensors
- Robust in real-world applications
- Best known approach to many hard robotics problems

Pitfalls

- Computationally demanding
- False assumptions
- Approximate

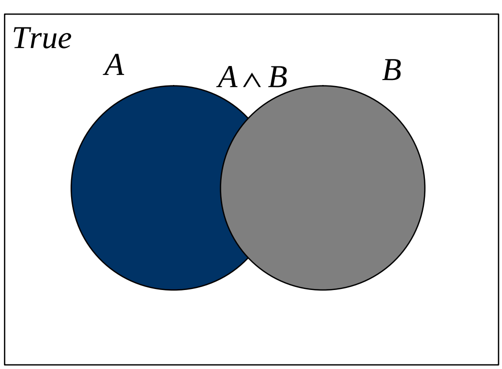
Axioms of Probability Theory

Pr*(A)* denotes probability that proposition *A* is true.

- $0 \le \Pr(A) \le 1$
- $\Pr(True) = 1 \qquad \Pr(False) = 0$
- $\Pr(A \vee B) = \Pr(A) + \Pr(B) - \Pr(A \wedge B)$

A Closer Look at Axiom 3

$$\Pr(A \vee B) = \Pr(A) + \Pr(B) - \Pr(A \wedge B)$$



Using the Axioms

$$\begin{aligned}
\Pr(A \vee \neg A) &= \Pr(A) + \Pr(\neg A) - \Pr(A \wedge \neg A) \\
\Pr(True) &= \Pr(A) + \Pr(\neg A) - \Pr(False) \\
1 &= \Pr(A) + \Pr(\neg A) - 0 \\
\Pr(\neg A) &= 1 - \Pr(A)
\end{aligned}$$

Discrete Random Variables

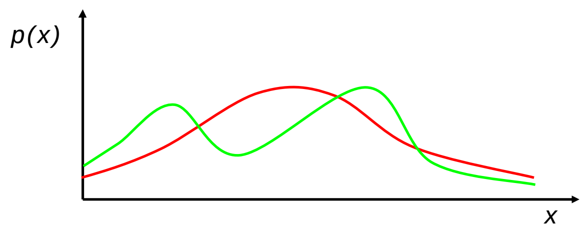
- X denotes a random variable.
- X can take on a finite number of values in $\{x_1, x_2, \ldots, x_n\}$.
- $P(X=x_i)$, or $P(x_i)$, is the probability that the random variable X takes on value x_i.
- $P(\cdot)$ is called probability mass function.
- E.g. $P(Room) = \langle 0.7, 0.2, 0.08, 0.02 \rangle$

Continuous Random Variables

- *X* takes on values in the continuum.
- *p(X=x)*, or *p(x)*, is a probability density function.

$$\Pr(x \in [a,b]) = \int_a^b p(x)dx$$

- E.g.



Joint and Conditional Probability

- $P(X=x \text{ and } Y=y) = P(x,y)$
- If X and Y are independent then
  $$P(x,y) = P(x)\,P(y)$$
- $P(x \mid y)$ is the probability of *x* given *y*
  $$P(x \mid y) = P(x,y) / P(y)$$
  $$P(x,y) = P(x \mid y)\,P(y)$$
- If X and Y are independent then
  $$P(x \mid y) = P(x)$$

Law of Total Probability, Marginals

Discrete case

$$\sum_x P(x) = 1$$

$$P(x) = \sum_y P(x, y)$$

$$P(x) = \sum_y P(x \mid y) P(y)$$

Continuous case

$$\int p(x)\, dx = 1$$

$$p(x) = \int p(x, y)\, dy$$

$$p(x) = \int p(x \mid y)\, p(y)\, dy$$

Bayes Formula

$$P(x, y) = P(x \mid y) P(y) = P(y \mid x) P(x)$$

$$\Rightarrow$$

$$P(x \mid y) = \frac{P(y \mid x)\ P(x)}{P(y)} = \frac{\text{likelihood} \cdot \text{prior}}{\text{evidence}}$$

Normalization

$$P(x|y) = \frac{P(y|x)\ P(x)}{P(y)} = h\ P(y|x)\,P(x)$$

$$h = P(y)^{-1} = \frac{1}{\sum_x P(y|x)P(x)}$$

Algorithm:

$$\forall x: \mathrm{aux}_{x|y} = P(y|x)\ P(x)$$

$$h = \frac{1}{\sum_x \mathrm{aux}_{x|y}}$$

$$\forall x: P(x|y) = h\ \mathrm{aux}_{x|y}$$

Conditioning

- Total probability:

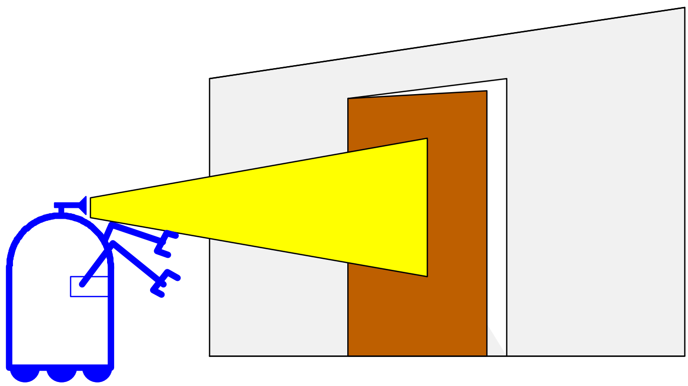
$$P(x|y) = \int P(x|y,z)\,P(z|y)\,dz$$

- Bayes rule and background knowledge:

$$P(x|y,z) = \frac{P(y|x,z)\ P(x|z)}{P(y|z)}$$

Simple Example of State Estimation

- Suppose a robot obtains measurement z
- What is *P(open|z)?*



Causal vs. Diagnostic Reasoning

- *P(open|z)* is diagnostic.
- *P(z|open)* is causal.
- Often causal knowledge is easier to obtain.
- Bayes rule allows us to use causal knowledge:

count frequencies!

$$P(open \mid z) = \frac{P(z \mid open)P(open)}{P(z)}$$

Example

- $P(z|open) = 0.6$ $P(z|\neg open) = 0.3$
- $P(open) = P(\neg open) = 0.5$

$$P(open \mid z) = \frac{P(z \mid open) P(open)}{P(z \mid open)\, p(open) + P(z \mid \neg open)\, p(\neg open)}$$

$$P(open \mid z) = \frac{0.6 \cdot 0.5}{0.6 \cdot 0.5 + 0.3 \cdot 0.5} = \frac{2}{3} = 0.67$$

- z raises the probability that the door is open.

Combining Evidence

- Suppose our robot obtains another observation z_2.
- How can we integrate this new information?
- More generally, how can we estimate $P(x| z_1 ... z_n)$?

Recursive Bayesian Updating

$$P(x \mid z_1,\ldots,z_n) = \frac{P(z_n \mid x, z_1,\ldots,z_{n-1})\, P(x \mid z_1,\ldots,z_{n-1})}{P(z_n \mid z_1,\ldots,z_{n-1})}$$

Markov assumption: z_n is independent of $z_1,\ldots,z_{n-1}$ if we know x.

$$P(x \mid z_1,\ldots,z_n) = \frac{P(z_n \mid x)\, P(x \mid z_1,\ldots,z_{n-1})}{P(z_n \mid z_1,\ldots,z_{n-1})}$$

$$= h\; P(z_n \mid x)\, P(x \mid z_1,\ldots,z_{n-1})$$

$$= h_{1\ldots n} \prod_{i=1\ldots n} P(z_i \mid x)\, P(x)$$

Example: Second Measurement

- $P(z_2|open) = 0.5$ $\qquad$ $P(z_2|\neg open) = 0.6$
- $P(open|z_1) = 2/3$

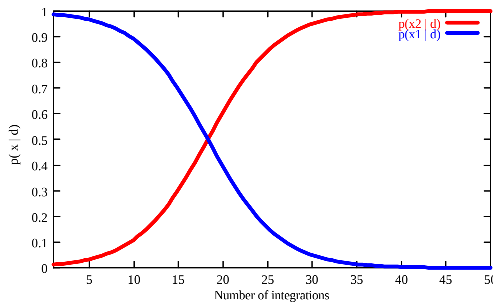
$$P(open \mid z_2, z_1) = \frac{P(z_2 \mid open)\, P(open \mid z_1)}{P(z_2 \mid open)\, P(open \mid z_1) + P(z_2 \mid \neg open)\, P(\neg open \mid z_1)}$$

$$= \frac{\frac{1}{2}\cdot\frac{2}{3}}{\frac{1}{2}\cdot\frac{2}{3}+\frac{3}{5}\cdot\frac{1}{3}} = \frac{5}{8} = 0.625$$

- z_2 lowers the probability that the door is open.

A Typical Pitfall

- Two possible locations x_1 and x_2
- $P(x_1)=0.99$
- $P(z|x_2)=0.09$ $P(z|x_1)=0.07$



Actions

- Often the world is **dynamic** since
  - **actions carried out by the robot**,
  - **actions carried out by other agents**,
  - or just the **time** passing by

  change the world.

- How can we **incorporate** such **actions**?

Typical Actions

- The robot **turns its wheels** to move
- The robot **uses its manipulator** to grasp an object
- Plants grow over **time**…

- Actions are **never carried out with absolute certainty**.
- In contrast to measurements, **actions generally increase the uncertainty**.

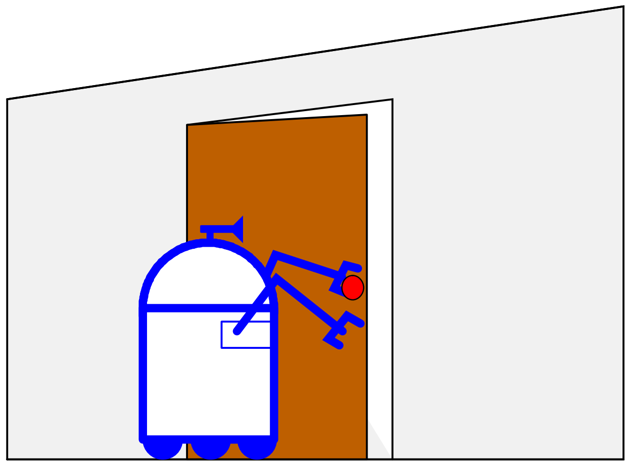
Modeling Actions

- To incorporate the outcome of an action *u* into the current "belief", we use the conditional pdf

$$P(x|u,x')$$

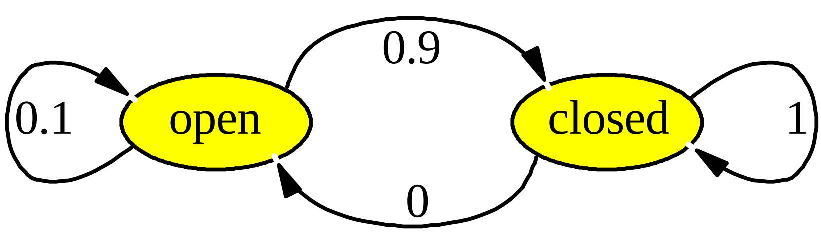
- This term specifies the pdf that **executing *u* changes the state from *x'* to *x*.**

Example: Closing the door



State Transitions

$P(x|u,x')$ for u = “close door”:



If the door is open, the action “close door” succeeds in 90% of all cases.

Integrating the Outcome of Actions

Continuous case:

$$P(x \mid u) = \int P(x \mid u, x')P(x')dx'$$

Discrete case:

$$P(x \mid u) = \sum P(x \mid u, x')P(x')$$

Example: The Resulting Belief

$$\begin{aligned}
P(closed \mid u) &= \sum P(closed \mid u, x')P(x') \\
&= P(closed \mid u, open)P(open) \\
&\quad + P(closed \mid u, closed)P(closed) \\
&= \frac{9}{10} * \frac{5}{8} + \frac{1}{1} * \frac{3}{8} = \frac{15}{16} \\
P(open \mid u) &= \sum P(open \mid u, x')P(x') \\
&= P(open \mid u, open)P(open) \\
&\quad + P(open \mid u, closed)P(closed) \\
&= \frac{1}{10} * \frac{5}{8} + \frac{0}{1} * \frac{3}{8} = \frac{1}{16} \\
&= 1 - P(closed \mid u)
\end{aligned}$$

Bayes Filters: Framework

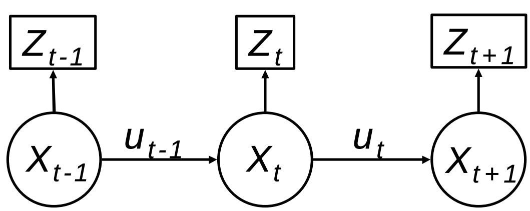
- **Given:**
  - Stream of observations *z* and action data *u:*

$$d_t = \{u_1, z_2 \ldots, u_{t-1}, z_t\}$$

  - Sensor model *P(z|x).*
  - Action model *P(x|u,x')*.
  - Prior probability of the system state *P(x).*
- **Wanted:**
  - Estimate of the state *X* of a dynamical system.
  - The posterior of the state is also called **Belief**:

$$Bel(x_t) = P(x_t \mid u_1, z_2 \ldots, u_{t-1}, z_t)$$

Markov Assumption



$$p(d_t, d_{t-1}, \ldots, d_0 \mid x_t, d_{t+1}, d_{t+2}, \ldots) = p(d_t, d_{t-1}, \ldots, d_0 \mid x_t)$$

$$p(d_t, d_{t+1}, \ldots \mid x_t, d_1, d_2, \ldots, d_{t-1}) = p(d_t, d_{t+1}, \ldots \mid x_t)$$

$$p(x_t \mid u_{t-1}, x_{t-1}, d_{t-2}, \ldots, d_0) = p(x_t \mid u_{t-1}, x_{t-1})$$

Underlying Assumptions

- Static world
- Independent noise
- Perfect model, no approximation errors

Bayes Filters

z = observation
u = action
x = state

$$Bel(x_t) = P(x_t \mid u_1, z_2 \ldots, u_{t-1}, z_t)$$

Bayes $$= h\ P(z_t \mid x_t, u_1, z_2, \ldots, u_{t-1})\ P(x_t \mid u_1, z_2, \ldots, u_{t-1})$$

Markov $$= h\ P(z_t \mid x_t)\ P(x_t \mid u_1, z_2, \ldots, u_{t-1})$$

Total prob. $$= h\ P(z_t \mid x_t) \int P(x_t \mid u_1, z_2, \ldots, u_{t-1}, x_{t-1})\ P(x_{t-1} \mid u_1, z_2, \ldots, u_{t-1})\ dx_{t-1}$$

Markov $$= h\ P(z_t \mid x_t) \int P(x_t \mid u_{t-1}, x_{t-1})\ P(x_{t-1} \mid u_1, z_2, \ldots, u_{t-1})\ dx_{t-1}$$

$$= h\ P(z_t \mid x_t) \int P(x_t \mid u_{t-1}, x_{t-1})\ Bel(x_{t-1})\ dx_{t-1}$$

$$Bel(x_t) = h\ P(z_t \mid x_t) \int P(x_t \mid u_{t-1}, x_{t-1})\ Bel(x_{t-1})\ dx_{t-1}$$

```
Algorithm Bayes_filter( Bel(x),d ):
h=0
if d is a perceptual data item z then
    For all x do
        Bel'(x) = P(z | x)Bel(x)
        h = h + Bel'(x)
    For all x do
        Bel'(x) = h^-1 Bel'(x)
else if d is an action data item u then
    For all x do
        Bel'(x) = ∫ P(x | u, x') Bel(x') dx'
return Bel'(x)
```

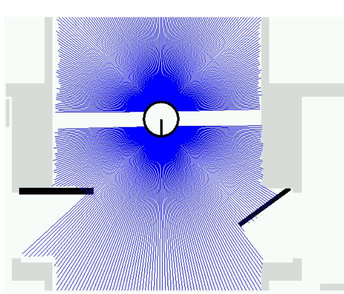
Bayes Filters are Familiar!

$$Bel(x_t) = \eta \; P(z_t \mid x_t) \int P(x_t \mid u_{t-1}, x_{t-1}) \; Bel(x_{t-1}) \; dx_{t-1}$$

- Kalman filters
- Particle filters
- Hidden Markov models
- Dynamic Bayes networks
- Partially Observable Markov Decision Processes (POMDPs)

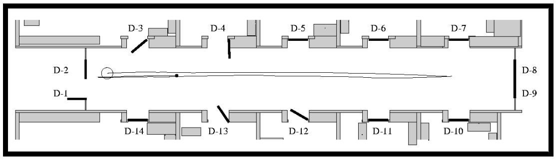
Application

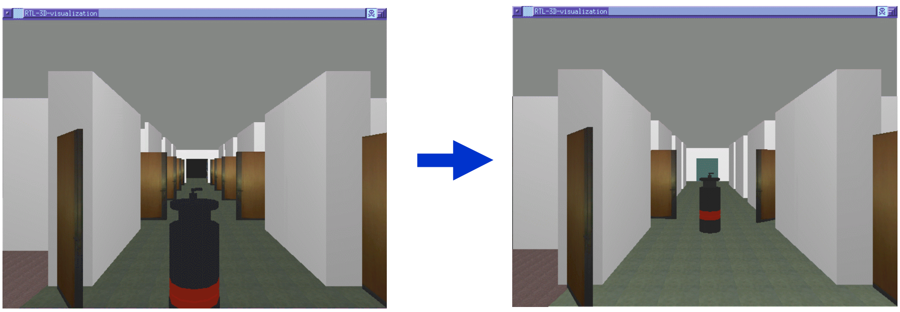
- Estimate the opening angle of a door
- and the state of other dynamic objects
- using a laser-range finder
- from a moving mobile robot and
- based on Bayes filters.



[Schulz et al., 01]

Result





[Schulz et al., 01]

Lessons Learned

- Bayes rule allows us to compute probabilities that are hard to assess otherwise.
- Under the Markov assumption, recursive Bayesian updating can be used to efficiently combine evidence.
- Bayes filters are a probabilistic tool for estimating the state of dynamic systems.