

Chapter 1

The Compressed Extended Kalman Filter (CEKF)

1.1 CEKF Definition

In this section an EKF modified version is presented. This filter is a particular case of the family of compressed filters that are presented in chapter (2). This filter takes advantage of the systems structure giving an estimation that is identical to the standard EKF but dramatically reducing the computational complexity.

This class of systems presents a state locality characteristic. Large sequences of predictions and observations steps are function of a reduced set of states (i.e. they are independent of most of the elements of the states vector). The CEKF takes advantage of this characteristic. The full estimation problem is temporarily converted to a reduced estimation process that finally gives a solution identical to the full EKF estimation. The CEKF can easily manage high frequency sequences of predictions and updates avoiding the expensive cost of the full EKF estimation approach.

Assume a discrete dynamic system

$$\begin{aligned} X(k+1) &= f(X(k), u(k), k) \\ X &\in R^n \end{aligned} \tag{1.1}$$

and an observational model

$$\begin{aligned} y(k) &= G(X(k), k) \\ y &\in R^m \end{aligned} \tag{1.2}$$

Consider the state vector divided into two groups of states (but not necessarily separately grouped)

$$\begin{aligned} X &= \begin{pmatrix} X_a \\ X_b \end{pmatrix} \\ X_a &\in R^{n_a}, \quad X_b \in R^{n_b}, \quad X \in R^n \end{aligned} \tag{1.3}$$

and the covariance matrix divided according to this states division

$$\begin{aligned} P &= \begin{pmatrix} P_{aa} & P_{ab} \\ P_{ba} & P_{bb} \end{pmatrix} = E\{(\bar{X} - X) \cdot (\bar{X} - X)^T\} \in R^{n \cdot n} \\ P_{aa} &= E\{(\bar{X}_a - X_a) \cdot (\bar{X}_a - X_a)^T\} \in R^{n_a \cdot n_a} \\ P_{bb} &= E\{(\bar{X}_b - X_b) \cdot (\bar{X}_b - X_b)^T\} \in R^{n_b \cdot n_b} \\ P_{ab} &= E\{(\bar{X}_a - X_a) \cdot (\bar{X}_b - X_b)^T\} \in R^{n_a \cdot n_b} \\ P_{ba} &= P_{ab}^T \in R^{n_b \cdot n_a} \end{aligned} \tag{1.4}$$

Suppose that for a time period $\Omega = \{k \mid k_1 \leq k \leq k_2\}$ the model and observations have the

following characteristic

$$\begin{aligned}
 \begin{pmatrix} X_a(k+1) \\ X_b(k+1) \end{pmatrix} &= \begin{pmatrix} f_a(X_a(k), u(k), k) + w_a(k) \\ X_b(k) + w_b(k) \end{pmatrix} \\
 y(k) &= h(X_a(k), k) + w_h(k) \\
 \forall (X, u, k) &/ k \in \Omega
 \end{aligned} \tag{1.5}$$

i.e. the set of states X_b does not have time evolution and the observation functions are independent of these states.

Model and observation noises are assumed Gaussian with the following characteristic

$$\begin{aligned}
 E\{w_a(k)\} &= \bar{0} \\
 E\{w_b(k)\} &= \bar{0} \\
 E\{w_h(k)\} &= \bar{0} \\
 E\{w_a(k) \cdot w_a(j)^T\} &= \delta_{k,j} \cdot Q_{aa}(k) \\
 E\{w_b(k) \cdot w_b(j)^T\} &= \delta_{k,j} \cdot Q_{bb}(k) \\
 E\{w_h(k) \cdot w_h(j)^T\} &= \delta_{k,j} \cdot R(k) \\
 E\{w_a(k) \cdot w_b(j)^T\} &= \bar{0} \\
 E\{w_a(k) \cdot w_h(j)^T\} &= \bar{0} \\
 E\{w_b(k) \cdot w_h(j)^T\} &= \bar{0}
 \end{aligned} \tag{1.6}$$

Then an EKF executing during this period of time, Ω , can be evaluated according to the "compressed EKF" structure as it is shown bellow.

At the beginning of the time period Ω (at k_1) a set of auxiliary matrixes, $\phi, \psi, \theta, Q_{bb}^*$, is created. These matrixes are never bigger than the P_{aa} (except Q_{bb}^* , but it is normally null or diagonal).

$$\begin{aligned}
\phi, \psi &\in R^{n_a \cdot n_a} \\
\theta &\in R^{n_a} \\
Q_{bb}^* &\in R^{n_b \cdot n_b}
\end{aligned} \tag{1.7}$$

having as initial condition at $k = k_1$

$$\phi(k_1) = I, \quad \psi(k_1) = \bar{\bar{0}}, \quad \theta(k_1) = \bar{0}, \quad Q_{bb}^*(k_1) = \bar{\bar{0}} \tag{1.8}$$

In every prediction or update stage, during the time period, a normal EKF is run over the subsystem X_a, P_{aa} and additional operations are done over the auxiliary matrixes. Then in any prediction done in the time period $k \setminus k_1 \leq k \leq k_2$ a standard EKF prediction step is done for X_a, P_{aa} and the auxiliary matrixes are updated according to

$$\begin{aligned}
\phi(k) &= J_{aa} \cdot \phi(k-1) \\
\psi(k) &= \psi(k-1) \\
\theta(k) &= \theta(k-1) \\
J_{aa} &= (\partial f_a / \partial X_a) |_{X_a(k)} \\
Q_{bb}^*(k) &= Q_{bb}^*(k-1) + Q_{bb}(k)
\end{aligned} \tag{1.9}$$

and in any update a standard EKF update step is done for the subsystem X_a, P_{aa} and the auxiliary matrixes are updated according to

$$\begin{aligned}
\phi(k) &= (I - \mu(k)) \cdot \phi(k-1) \\
\psi(k) &= \psi(k-1) + \phi^T(k-1) \cdot \beta(k) \cdot \phi(k-1) \\
\theta(k) &= \theta(k-1) + \phi^T(k-2) \cdot H_a^T(k-1) \cdot S(k-1)^{-1} \cdot z(k-1) \\
H_a(k) &= (\partial h / \partial X_a)|_{X_a(k)} \\
\beta(k) &= H_a^T(k) \cdot S(k)^{-1} \cdot H_a(k) \\
\mu_k &= P_{aa}(k) \cdot \beta(k) \\
S(k) &= H_a(k) \cdot P_{aa}(k) \cdot H_a^T(k) + R \\
z(k) &= y(k) - h(X_a(k), k)
\end{aligned} \tag{1.10}$$

At the end of the interval Ω (at k_2) the update of all the states and the complete covariance matrix is done. This step is called "global update"

$$\begin{aligned}
P_{ab}(k_2) &= \phi(k_2) \cdot P_{ab}(k_1) \\
P_{bb}(k_2) &= P_{bb}(k_1) - P_{ab}(k_1) \cdot \psi(k_2) \cdot P_{ab}(k_1) + Q_{bb}^*(k_2) \\
X_b(k_2) &= X_b(k_1) - P_{ab}(k_1) \cdot \theta(k_2)
\end{aligned} \tag{1.11}$$

It can be seen that the knowledge of $X_b(k), P_{bb}(k), P_{ba}(k), P_{ab}(k)$ is only explicit in times $k = k_1$ and $k = k_2$. No explicit information about this family of states and their related covariance and cross-covariance is known in $k \setminus k_1 < k < k_2$. All this information is implicitly contained (at any instant k) in the auxiliary matrixes $\phi(k), \psi(k), \theta(k)$. The matrixes $\phi(k), \psi(k)$ have dimension $n_a \cdot n_a$ and $\theta(k)$ is a column vector of length n_a . All the information (covariance, cross-covariance, value) of the set of states $X_b(k)$ is compressed in these 3 auxiliary matrices.

1.2 Demonstration

In any EKF update stage done in $k \setminus k_1 \leq k \leq k_2$ the Jacobian matrix, H , of the observation function will have the following structure

$$H = H(k) = (\partial G / \partial X)|_{X(k)} = (\partial G / \partial (X_a, X_b))|_{X(k)} = \begin{pmatrix} \partial G / \partial X_a & \partial G / \partial X_b \end{pmatrix} = \begin{pmatrix} H_a & 0 \end{pmatrix} \quad (1.12)$$

From the Jacobian matrix H it can be seen that at least all the columns related to the states X_b are zero and the evaluation of the sub-matrix H_a is independent of the states X_b . If the update step is analyzed dividing the covariance matrix in sub-matrixes representing the covariance of the states X_a, X_b and the cross-covariance between these two families of states, as in (1.4), then the intermediate matrixes used in the EKF update step can be divided

$$P \cdot H^T = P \cdot \begin{pmatrix} H_a & \bar{0} \end{pmatrix}^T = \begin{pmatrix} P_{aa} & P_{ab} \\ P_{ba} & P_{bb} \end{pmatrix} \cdot \begin{pmatrix} H_a^T \\ \bar{0} \end{pmatrix} = \begin{pmatrix} P_{aa} \cdot H_a^T \\ P_{ba} \cdot H_a^T \end{pmatrix} \quad (1.13)$$

$$H \cdot P \cdot H^T = \begin{pmatrix} H_a & \bar{0} \end{pmatrix} \cdot \begin{pmatrix} P_{aa} \cdot H_a^T \\ P_{ba} \cdot H_a^T \end{pmatrix} = H_a \cdot P_{aa} \cdot H_a^T$$

Then $h_k = h(X_a(k), k)$, that is a function independent of the random variable $X_b(k)$, has covariance

$$E\{(\bar{h}_k - h_k) \cdot (\bar{h}_k - h_k)^T\} = H(k) \cdot P(k) \cdot H^T(k) = H_a(k) \cdot P_{aa}(k) \cdot H_a^T(k) \quad (1.14)$$

The innovation sequence $z_k = y(k) - h_k$ has covariance

$$E\{(\bar{z}_k - z_k) \cdot (\bar{z}_k - z_k)^T\} = S(k) = H_a(k) \cdot P_{aa}(k) \cdot H_a^T(k) + R(k) \quad (1.15)$$

where

$$R(k) = E\{(\overline{y(k)} - y(k)) \cdot (\overline{y(k)} - y(k))^T\} \quad (1.16)$$

The gain matrix

$$W(k) = P(k) \cdot H^T(k) \cdot S(k)^{-1} = \begin{pmatrix} P_{aa}(k) \cdot H_a^T(k) \cdot S(k)^{-1} \\ P_{ba}(k) \cdot H_a^T(k) \cdot S(k)^{-1} \end{pmatrix} = \begin{pmatrix} W_a(k) \\ W_b(k) \end{pmatrix} \quad (1.17)$$

From these equations it is possible to see that

- The Jacobian matrix H_a has no dependence on the states X_b .
- The S and W_a matrixes are function of P_{aa} and H_a but they do not have dependence on P_{bb} , P_{ab} , P_{ba} and X_b .

The covariance matrix change after one update is here analyzed

$$\begin{aligned} dP &= \begin{pmatrix} (P_{aa} \cdot H_a^T \cdot S^{-1} \cdot H_a \cdot P_{aa}) & (P_{aa} \cdot H_a^T \cdot S^{-1} \cdot H_a \cdot P_{ab}) \\ (P_{aa} \cdot H_a^T \cdot S^{-1} \cdot H_a \cdot P_{ab})^T & (P_{ba} \cdot H_a^T \cdot S^{-1} \cdot H_a \cdot P_{ab}) \end{pmatrix} = \\ &= \begin{pmatrix} P_{aa} \cdot \beta \cdot P_{aa} & \mu \cdot P_{ab} \\ dP_{ab}^T & P_{ba} \cdot \beta \cdot P_{ab} \end{pmatrix} \\ &\quad \beta = H_a^T \cdot S^{-1} \cdot H_a \\ &\quad \mu = P_{aa} \cdot \beta \end{aligned} \quad (1.18)$$

It is possible to see that after one update the covariance matrix is (for simplicity it is assumed $k_1 = 0$)

$$P(1) = P(0) - dP(0)$$

$$P_{aa}(1) = P_{aa}(0) - P_{aa}(0) \cdot \beta(0) \cdot P_{aa}(0) \quad (1.19)$$

$$P_{ab}(1) = P_{ab}(0) - \mu(0) \cdot P_{ab}(0) = (I - \mu(0)) \cdot P_{ab}(0)$$

$$P_{bb}(1) = P_{bb}(0) - P_{ba}(0) \cdot \beta(0) \cdot P_{ab}(0)$$

After k consecutive updates the change in the covariance sub-matrices P_{bb} , P_{ab} , P_{ba} is

$$P_{ab}(k) = \phi(k-1) \cdot P_{ab}(0)$$

$$P_{bb}(k) = P_{bb}(0) - P_{ba}(0) \cdot \psi(k-1) \cdot P_{ab}(0)$$

$$\phi(k) = (I - \mu(k)) \cdot (I - \mu(k-1)) \cdot \dots \cdot (I - \mu(0)) \quad (1.20)$$

$$\psi(k) = \sum_{i=1}^k (\phi^T(i-1) \cdot \beta(i) \cdot \phi(i-1)) + \beta(0)$$

$$\beta(i) = H_a^T(i) \cdot S^{-1}(i) \cdot H_a(i)$$

$$\mu(i) = P_{aa}(i) \cdot \beta(i)$$

$$H_a(i) = (\partial h(X_a) / \partial X_a) |_{X_a(i)}$$

The calculation of the $\phi(k)$, $\psi(k)$ matrixes can be done recursively according to:

$$\phi(k) = (I - \mu(k)) \cdot \phi(k-1) \quad (1.21)$$

$$\psi(k) = \psi(k-1) + \phi^T(k-1) \cdot \beta(k) \cdot \phi(k-1)$$

with initial conditions

$$\phi(0) = I - \mu(0) \quad (1.22)$$

$$\psi(0) = \beta(0)$$

The states X_b can be updated after k update steps using the following equations

$$X_b(k) = X_b(0) - P_{ba}(0) \cdot \theta(k) \quad (1.23)$$

where the auxiliary matrix $\theta(k)$ is recursively calculated according to

$$\begin{aligned} \theta(k) &= \theta(k-1) + \phi^T(k-1) \cdot S^{-1}(k-1) \cdot z(k-1) \\ \theta(0) &= \bar{0} \end{aligned} \quad (1.24)$$

This expression is derived from:

$$X^+ = X^- + dX$$

$$dX = W \cdot (h(X^-) - Y) = W \cdot z$$

$$z = h(X_a^-) - Y \quad (1.25)$$

$$W = \begin{pmatrix} W_a \\ W_b \end{pmatrix}$$

$$X_b^+ = X_b^- - W_b \cdot z$$

and

$$X_b(1) = X_b(0) - W_b(0) \cdot z(0)$$

$$\begin{aligned} X_b(k) &= X_b(0) - \sum_{i=0}^{k-1} W_b(i) \cdot z(i) = \\ &= X_b(0) - \sum_{i=0}^{k-1} \phi^T(i-1) \cdot H_a^T(i) \cdot S(i)^{-1} \cdot z(i) \end{aligned} \tag{1.26}$$

$$X_b(k) = X_b(0) - P_{ba}(0) \cdot \theta(k)$$

$$\begin{aligned} \theta(k) &= \sum_{i=0}^{k-1} \phi^T(i-1) \cdot H_a^T(i) \cdot S(i)^{-1} \cdot z_i = \\ &= \theta(k-1) + \phi^T(k-2) \cdot H_a^T(k-1) \cdot S(k-1)^{-1} \cdot z(k-1) \end{aligned}$$

Similar results are obtained for sequences of interleaved prediction and update steps, considering that those updates and predictions involve functions that are independent of the states X_b . In the prediction step, the model and noise are assumed to be as in (1.5) and (1.6), then a prediction step will be

$$P(1) = J \cdot P(0) \cdot J^T + Q$$

$$\begin{aligned} J &= \begin{pmatrix} J_{aa} & J_{ab} \\ J_{ba} & J_{bb} \end{pmatrix} = \begin{pmatrix} J_{aa} & \bar{0} \\ \bar{0} & I \end{pmatrix} \\ Q &= \begin{pmatrix} Q_{aa} & \bar{0} \\ \bar{0} & Q_{bb} \end{pmatrix} \end{aligned} \tag{1.27}$$

$$P_{aa}(1) = J_{aa} \cdot P_{aa}(0) \cdot J_{aa}^T + Q_{aa}$$

$$P_{ab}(1) = J_{aa} \cdot P_{ab}(0)$$

$$P_{ba}(1) = (P_{ab}(1))^T$$

$$P_{bb}(1) = P_{bb}(0) + Q_{bb}$$

Then for any sequence of interleaved predictions and updates the following will result:

if it is a prediction step: a standard EKF prediction is done over the subsystem X_a , P_{aa} and the auxiliary matrixes are modified according to

$$\begin{aligned}
 \phi(k) &= J_{aa} \cdot \phi(k-1) \\
 \psi(k) &= \psi(k-1) \\
 \theta(k) &= \theta(k-1) \\
 Q_{bb}(k) &= Q_{bb}^*(k-1) + Q_{bb}(k)
 \end{aligned} \tag{1.28}$$

In the case of an update step: a standard EKF update is done over the subsystem X_a , P_{aa} and the auxiliary matrixes are modified according to

$$\begin{aligned}
 \phi(k) &= (I - \mu(k)) \cdot \phi(k-1) \\
 \psi(k) &= \psi(k-1) + \phi(k-1)^T \cdot \beta(k) \cdot \phi(k-1) \\
 \theta(k) &= \theta(k-1) + \phi^T(k-2) \cdot H_a^T(k-1) \cdot S(k-1)^{-1} \cdot z(k-1)
 \end{aligned} \tag{1.29}$$

1.3 Appending new states

The dimension of the system can increase at any time however this situation does not produce any additional complexity to the CEKF.

Suppose that in time $k^* \setminus k_1 \leq k^* \leq k_2$ a new group of states, $X_{new} \in R^{n_{new}}$, is appended

$$X^+ = \begin{pmatrix} X^- \\ X_{new} \end{pmatrix} \tag{1.30}$$

where X^- is the system states vector before the expansion is done and X^+ is the expanded system states vector. After that, the expanded system evolves having an initial condition in $k = k^*$ and subsequent dynamic for $k > k^*$ as

$$\begin{aligned} X^+(k^*) &= \begin{pmatrix} X^-(k^*) \\ X_{new}^0 \end{pmatrix} \\ \begin{pmatrix} X_a(k+1) \\ X_{new}(k+1) \\ X_b(k+1) \end{pmatrix} &= \begin{pmatrix} f_a(X_a(k), X_{new}(k), u(k), k) \\ f_{new}(X_a(k), X_{new}(k), u(k), k) \\ X_b(k) \end{pmatrix} \end{aligned} \quad (1.31)$$

$$y = h(X_a, X_{new}, k)$$

$$(X, u, k) \setminus k^* \leq k < k_2$$

Reordering the states according to

$$X_a^{new} = \begin{pmatrix} X_a \\ X_{new} \end{pmatrix}, X_b^{new} = X_b \quad (1.32)$$

and expressing the system in a more compact form

$$\begin{pmatrix} X_a^{new}(k+1) \\ X_b(k+1) \end{pmatrix} = \begin{pmatrix} f_a^{new}(X_a^{new}(k), u(k), k) \\ X_b(k) \end{pmatrix} \quad (1.33)$$

$$y = h(X_a^{new}, k)$$

$$(X, u, k) \setminus k^* \leq k < k_2$$

it can be seen that the system has a structure similar to (1.5)

A trivial strategy to manage this change in the system structure is based on the division of the time interval Ω into two intervals Ω^- and Ω^+ ,

$$\begin{aligned}\Omega^- &= k \setminus k_1 \leq k \leq k^* \\ \Omega^+ &= k \setminus k^* \leq k \leq k_2\end{aligned}\tag{1.34}$$

and the performing of a global update at the intervals transition, in time k^* , but this global update can be avoided. The change in the dimension of the system can be managed by making minor changes to the initial conditions (related to the beginning of the time period, k_1) $P_{ab}(k_1)$ and to the auxiliary matrixes $\phi(k), \psi(k), \theta(k)$. These changes are shown in the following equation

$$\begin{aligned}P_{aa}^{new}(k^*) &= \begin{pmatrix} P_{aa}(k^*) & \bar{0} \\ \bar{0} & M \end{pmatrix}, \\ P_{aa} &\in R^{n_a \cdot n_a}, M = m \cdot I, m \longrightarrow \infty \\ P_{bb}^{new}(k_1) &= P_{bb}(k_1) \\ P_{ab}^{new}(k_1) &= \begin{pmatrix} P_{ab}(k_1) \\ \bar{0} \end{pmatrix}, \phi^{new}(k^*) = \begin{pmatrix} \phi(k^*) & \bar{0} \\ \bar{0} & I \end{pmatrix}, \psi^{new}(k^*) = \begin{pmatrix} \psi(k^*) & \bar{0} \\ \bar{0} & \bar{0} \end{pmatrix}\end{aligned}\tag{1.35}$$

In the first update involving the new states X_{new} the limit of these expressions having $m \longrightarrow \infty$ is solved and the correct initialization of $P_{aa}(k), \phi(k), \psi(k), \theta(k)$ is obtained. The initialization using the unbounded diagonal sub-matrix M is based on assuming of the existence of the new states but considering them completely uncertain before the first observation involving these new states is carried out. The evaluation cost of this procedure is similar to the cost of a normal compressed update.

1.4 A more general system structure

A more general case can be considered. Suppose a system with prediction and observational models that have the following characteristic

$$\begin{aligned} X(k+1) &= f(x(k), u(k), k) \\ y(k) &= G(X(k), k) \end{aligned} \tag{1.36}$$

$$(X, u, k) \setminus k \in \Omega$$

$$\Omega = \{k \setminus k_1 \leq k \leq k_2\}$$

and being

$$\Omega = \bigcup_{i=1}^J \Omega_i$$

$$\Omega_i = \{k \setminus k_{1,i} \leq k \leq k_{2,i}\} \tag{1.37}$$

$$k_{2,i} = k_{1,i+1}$$

with the system having the following property

$$\begin{aligned} X &= \begin{pmatrix} X_{a_i} \\ X_{b_i} \end{pmatrix} \\ \begin{pmatrix} X_{a_i}(k+1) \\ X_{b_i}(k+1) \end{pmatrix} &= \begin{pmatrix} f_i(X_{a_i}(k), u(k), k) \\ X_{b_i}(k) \end{pmatrix} \end{aligned} \tag{1.38}$$

$$y(k) = h_i(X_{a_i}(k), k)$$

$$(X, u, k) \setminus k \in \Omega_i$$

$$X_{a_i} \in R^{n_{a_i}}, X_{b_i} \in R^{n_{b_i}}$$

where the functions $f_i()$ and $h_i()$ do not need to be continuous with respect to the time variable, k .

Definition (1.38) means that for a period of time Ω_i the states can be grouped in an active states group X_{a_i} and in a passive states group X_{b_i} and the system will belong to the family (1.5). Under these conditions an EKF estimator can be run as a compressed EKF in each period Ω_i , performing a global update in each transition t_i between periods Ω_i and Ω_{i+1} .

$$\begin{aligned} t_i &= k_{i,2} = k_{i+1,1} \\ \Omega_i &= [k_{i,1}, k_{i,2}], \quad \Omega_{i+1} = [k_{i+1,1}, k_{i+1,2}] \end{aligned} \tag{1.39}$$

1.5 Summary

This chapter has presented the Compressed EKF estimator. This filter takes advantage of a particular characteristic that is present in a family of systems. The SLAM estimation problem presents this characteristic. The filter produces an estimate that is identical to the EKF estimate but its computational cost can be remarkably lower. The CEKF filter allows the estimation process to manage in real time high frequency sequences of predictions and updates. A full EKF estimator could not operate in similar conditions.

Chapter 2

General compressed filters

In this section a more general compressed filter vision is discussed. Similarly to the CEKF these filters take advantage of the systems structure. The CEKF version can be understood as a particular case of the compressed filter family.

As was seen in chapter (1) the CEKF obtains exactly the same result as the full EKF. It means that unsatisfactory EKF results will imply unsatisfactory CEKF results. In some estimation problems where the EKF cannot perform adequately other estimators can be applied. For instance "Second Order" filters, the "Unscented Filter", etc. Unfortunately these estimators can be too expensive to be applied in high scale estimation cases.

In systems with state locality characteristics (for a period of time the observations and predictions do not involve part of the states vector), as was defined in (1.36),(1.37) and (1.38), a combination of the EKF and other more powerful estimation approaches can be used.

In this hybrid method the EKF component is used to perform the global updates (similarly to the CEKF). The internal (local) compressed operation can be done by any type of estimation technique.

The non hybrid compressed filter, the CEKF, applies an EKF to perform the internal operation. It is a particular type of compressed filter.

2.1 Definition

Suppose a discrete dynamic system as in (1.5).

The states are classified in 2 groups.

$$X = \begin{pmatrix} X_a \\ X_b \end{pmatrix} \quad (2.1)$$

$$X_a \in R^{n_a}, \quad X_b \in R^{n_b}, \quad X \in R^n$$

and the covariance matrix divided according to this states division.

$$\begin{aligned} P &= \begin{pmatrix} P_a & P_b \end{pmatrix} = \begin{pmatrix} P_{aa} & P_{ab} \\ P_{ba} & P_{bb} \end{pmatrix} = E\{(\bar{X} - X) \cdot (\bar{X} - X)^T\} \in R^{n \cdot n} \\ P_a &= E\{(\bar{X}_a - X_a) \cdot (\bar{X} - X)^T\} \in R^{n_a \cdot n} \\ P_b &= E\{(\bar{X}_b - X_b) \cdot (\bar{X} - X)^T\} \in R^{n_b \cdot n} \\ P_{aa} &= E\{(\bar{X}_a - X_a) \cdot (\bar{X}_a - X_a)^T\} \in R^{n_a \cdot n_a} \\ P_{bb} &= E\{(\bar{X}_b - X_b) \cdot (\bar{X}_b - X_b)^T\} \in R^{n_b \cdot n_b} \\ P_{ab} &= E\{(\bar{X}_a - X_a) \cdot (\bar{X}_b - X_b)^T\} \in R^{n_a \cdot n_b} \\ P_{ba} &= P_{ab}^T \in R^{n_b \cdot n_a} \end{aligned} \quad (2.2)$$

For a time period $\Omega = \{k \mid k_1 \leq k \leq k_2\}$ the model and observations have the following characteristics

$$\begin{aligned} \begin{pmatrix} X_a(k+1) \\ X_b(k+1) \end{pmatrix} &= \begin{pmatrix} f_a(X_a(k), u(k), k) + w_a(k) \\ X_b(k) + w_b(k) \end{pmatrix} \\ y(k) &= h(X_a(k), k) + w_h(k) \\ \forall (X, u, k) / k &\in \Omega \end{aligned} \quad (2.3)$$

Model and observation have Gaussian noises according to

$$\begin{aligned}
E\{w_a(k)\} &= \bar{0} \\
E\{w_b(k)\} &= \bar{0} \\
E\{w_h(k)\} &= \bar{0} \\
E\{w_a(k) \cdot w_a(j)^T\} &= \delta_{k,j} \cdot Q_{aa}(k) \\
E\{w_b(k) \cdot w_b(j)^T\} &= \delta_{k,j} \cdot Q_{bb}(k) \\
E\{w_h(k) \cdot w_h(j)^T\} &= \delta_{k,j} \cdot R(k) \\
E\{w_a(k) \cdot w_b(j)^T\} &= \bar{0} \\
E\{w_a(k) \cdot w_h(j)^T\} &= \bar{0} \\
E\{w_b(k) \cdot w_h(j)^T\} &= \bar{0}
\end{aligned} \tag{2.4}$$

Then for the time period Ω a reduced version of the full system can be considered, ignoring for this time period any (indirect) update over the rest of the system state estimates and associated covariances.

First the original system is augmented in n_a states, X_{a2} .

$$X = \begin{pmatrix} X_{a2} \\ X_a \\ X_b \end{pmatrix} \tag{2.5}$$

$$X_{a2}(k_1) = X_a(k_1)$$

and its covariance matrix is initialized according to

$$P = \begin{pmatrix} m \cdot \bar{I} & \bar{0} & \bar{0} \\ \bar{0} & P_{aa}(k_1) & P_{ab}(k_1) \\ \bar{0} & P_{ba}(k_1) & P_{bb}(k_1) \end{pmatrix} \tag{2.6}$$

$$m \longrightarrow \infty$$

where complete uncertainty with respect to the states X_{a2} is assumed. This augmented system is frozen during the time interval $[k_1, k_2]$. A reduced system is executed during this time period. The system represents the subsystem $\{X_a, X_{a2}\}$ (called $\{X_A\}$) of the full system.

$$\begin{aligned}
 y &= \begin{pmatrix} y_2 \\ y_1 \end{pmatrix} \\
 y_1(k+1) &= y_1(k) \\
 y_2(k+1) &= f_a(y_2(k), u(k)) \\
 y_1(k_1) &= X_a(k_1) \\
 y_2(k_1) &= X_a(k_1)
 \end{aligned} \tag{2.7}$$

and initial covariance matrix

$$\begin{aligned}
 P_y(k_1|k_1) &= \begin{pmatrix} m \cdot \bar{I} & \bar{0} \\ \bar{0} & P_{aa}(k_1) \end{pmatrix} \\
 m &\longrightarrow \infty
 \end{aligned} \tag{2.8}$$

This reduced system represents the subsystem $\{X_a, X_{a2}\}$. It considers the dynamic evolution and the observations related to the subsystem X_A .

All the observations involving the states $X_a(k)$, $h(X_a(k)) = \bar{0}$, are applied as observations over the reduced system as $h(y_2(k)) = \bar{0}$.

Initially a virtual perfect observation is done. It is $y_2(k_1) = y_1(k_1)$. This observation forces the reduced system to assume that the two set of states y_1 and y_2 are identical at the initial time k_1 . This observation produces a new covariance matrix that is completely correlated.

$$P_y(k_1)^+ = \begin{pmatrix} P_{aa}(k_1) & P_{aa}(k_1) \\ P_{aa}(k_1) & P_{aa}(k_1) \end{pmatrix} \quad (2.9)$$

After this step a sequence of observations and predictions is carried out. Any observation involving $X_a(k)$ will be assumed as a function of $y_2(k)$.

$$h_i(X_a(k)) = \bar{0} \Rightarrow h_i(y_2(k)) = \bar{0} \quad (2.10)$$

Similarly any prediction step

$$X_a(k+1|k) = f_a(X_a(k|k), u(k), k) \Rightarrow y_2(k+1|k) = f_a(y_2(k|k), u(k), k) \quad (2.11)$$

At the end of the time period, at k_2 , the following inequality will be true

$$P_y(k_1|k_1) \geq P_y(k_2|k_2) \quad (2.12)$$

This assumption is always valid provided that

$$P_{y_2, y_2}(k_1|k_1) = m \cdot \bar{I} \quad (2.13)$$

$$m \longrightarrow \infty$$

and

$$P_{y_1, y_1}(k_1|k_1) \geq P_{y_1, y_1}(k_2|k_2) \quad (2.14)$$

equation (2.14) is always valid since the states $y_1(k)$ do not change (perfect invariant model $y_1(k+1|k) = y_1(k|k)$). Their estimated quality can only improve (or in the worst case it does not change). More exactly, they cannot lose information. The improvement is due to the information coming from the observations involving the states $y_2(k)$ that are correlated to them.

Then (2.12) is valid provided that (2.13) and (2.14) are valid.

This is a particular case (with $k \rightarrow \infty$) of the rule

$$\begin{aligned} & \begin{pmatrix} A & \xi \\ \xi^T & B \end{pmatrix} \geq \bar{0} \\ & \quad \downarrow \\ & \begin{pmatrix} (1+k) \cdot A & \bar{0} \\ \bar{0} & (1+1/k) \cdot B \end{pmatrix} \cong \begin{pmatrix} A & \xi \\ \xi^T & B \end{pmatrix} \\ & \quad \forall k > 0 \end{aligned} \tag{2.15}$$

The condition (2.12) enables the creation of a virtual observation over the full augmented system :

$$\begin{aligned} H_{virtual}(X_A) &= \bar{0} \\ X_A &= \begin{pmatrix} X_a \\ X_{a2} \end{pmatrix} \end{aligned} \tag{2.16}$$

This virtual observation has the following effect over the full system

$$\begin{aligned} P_{X_A, X_A}(k_2|k_2) &= P_{y,y}(k_2|k_2) \\ X_A(k_2|k_2) &= y(k_2|k_2) \end{aligned} \tag{2.17}$$

and simultaneously improves the complete covariance matrix and states of the rest of the full system, $P_{Ab}(k_2|k_2)$, $P_{bA}(k_2|k_2)$, $P_{bb}(k_2|k_2)$, $X_b(k_2|k_2)$.

It should be noted that the estimation process performed over the reduced subsystem can be done using any estimation technique provided that it can be finally expressed (at time k_2) as a Gaussian probability distribution. This Gaussian probability distribution defines its mean value (the estimated values, $y(k_2|k_2)$) and its covariance matrix ($P_y(k_2|k_2)$).

The virtual update over the full system is an EKF update.

For instance an UF (Unscented Filter) can be applied over the reduced system during the time period $[k_1, k_2]$. Then a better performance with respect to a pure EKF filter is obtained.

2.2 The virtual update

The objective of the virtual update (*v.u.*, or virtual observation, *v.o.*) is to transfer the collected information (during the time interval $[k_1, k_2]$) coming from the reduced subsystem (that was optimally estimated) to the full system. It means that after the *v.u.* part of the full system will be a copy of the reduced system. That situation is true provided that the reduced system has been representing the variant and directly observed part of the full system. The *v.u.* will have the form:

$$h(X_a) = X_a = \xi \tag{2.18}$$

where ξ is unknown and it does not need to be known. Similarly the quality of this *v.o.* does not need to be calculated, it is implicitly used but never explicitly calculated.

The improvement over the subsystem X_A (that is part of the full system) is obtained from the reduced system. It is the covariance change ΔP_{AA} and the states X_A change, called

ΔX_A .

$$\begin{aligned}\Delta P_{AA} &= P_y(k_1|k_1) - P_y(k_2|k_2) \\ \Delta X_A &= y(k_1|k_1) - y(k_2|k_2)\end{aligned}\tag{2.19}$$

The *v.o.* coming from (2.18) have a Jacobian matrix

$$\partial h(X)/\partial X = \begin{pmatrix} \partial h(X_A)/\partial X_A & \bar{0} \end{pmatrix} = \begin{pmatrix} H_A & \bar{0} \end{pmatrix} = \begin{pmatrix} \bar{I} & \bar{0} \end{pmatrix}\tag{2.20}$$

then

$$\begin{aligned}\Delta P_{AA} &= P_y(k_1|k_1) - P_y(k_2|k_2) \\ \Delta X_A &= y(k_2|k_2) - y(k_1|k_1)\end{aligned}$$

$$\mu = (P_{AA}(k_1|k_1))^{-1} \cdot P_{Ab}(k_1|k_1)\tag{2.21}$$

$$\begin{aligned}\Delta P_{Ab} &= \Delta P_{AA} \cdot \mu \\ \Delta P_{bb} &= \mu^T \cdot \Delta P_{AA} \cdot \mu\end{aligned}$$

$$\Delta X_b = \mu^T \cdot \Delta X_A$$

It comes from

$$\begin{aligned}
S &= H \cdot P(k_1|k_1) \cdot H^T + R = H_A \cdot P_{AA}(k_1|k_1) \cdot H_A^T + R \\
\Delta P_{AA} &= W_A \cdot S^{-1} \cdot W_A^T \\
W_A &= P_A(k_1|k_1) \cdot H^T = P_{AA}(k_1|k_1) \cdot H_A^T = P_{AA}(k_1|k_1) \\
W_b &= P_b(k_1|k_1) \cdot H^T = P_{bA}(k_1|k_1) \cdot H_A^T = P_{bA}(k_1|k_1) \\
\Delta P_{Ab} &= W_A \cdot S^{-1} \cdot W_b^T \\
&= \Delta P_{AA} \cdot (P_{AA}(k_1|k_1))^{-1} \cdot P_{Ab}(k_1|k_1) = \Delta P_{AA} \cdot \mu
\end{aligned} \tag{2.22}$$

$$\begin{aligned}
\Delta P_{bb} &= W_b \cdot S^{-1} \cdot W_b^T \\
&= \mu^T \cdot \Delta P_{AA} \cdot \mu
\end{aligned}$$

$$\begin{aligned}
\Delta X_A &= y(k_2) - y(k_1) \\
\Delta X_A &= K_A \cdot \xi = W_A \cdot S^{-1} \cdot \xi \\
\Delta X_b &= K_b \cdot \xi \\
&= W_b \cdot S^{-1} \cdot \xi = W_b \cdot W_A^{-1} \cdot \Delta X_A = P_{bA}(k_1|k_1) \cdot P_{AA}(k_1|k_1)^{-1} \cdot \Delta X_A
\end{aligned}$$

After the *v.u.* is done the frozen states X_a can be deleted from the system states X . The states X_{a2} are now the ones that represent the active states at time k_2 . The full system becomes

$$X = \begin{pmatrix} X_{a2} \\ X_b \end{pmatrix} \tag{2.23}$$

For the next time period $[k_2, k_3]$ the set of states X_{a2} can be named X_a .

2.2.1 Considering some implicit simplifications

Some simplifications can be done in the previous calculations to avoid unnecessary calculation. These simplifications are related to the matrix component that is initially unbounded.

$$P_y(k_1|k_1)^{-1} = \begin{pmatrix} m \cdot \bar{I} & \bar{0} \\ \bar{0} & P_{aa}(k_1|k_1) \end{pmatrix}^{-1} = \begin{pmatrix} 1/m \cdot \bar{I} & \bar{0} \\ \bar{0} & P_{aa}(k_1|k_1)^{-1} \end{pmatrix} \xrightarrow{m \rightarrow \infty} \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{0} & P_{aa}(k_1|k_1)^{-1} \end{pmatrix} \quad (2.24)$$

$$P_{Ab}(k_1|k_1) = \begin{pmatrix} P_{a_2b}(k_1|k_1) \\ P_{ab}(k_1|k_1) \end{pmatrix} = \begin{pmatrix} \bar{0} \\ P_{ab}(k_1|k_1) \end{pmatrix}$$

$$\begin{aligned} \mu = P_{AA}(k_1|k_1)^{-1} \cdot P_{Ab}(k_1|k_1) &= \begin{pmatrix} 1/m \cdot \bar{I} & \bar{0} \\ \bar{0} & P_{aa}(k_1|k_1)^{-1} \end{pmatrix} \cdot \begin{pmatrix} \bar{0} \\ P_{ab}(k_1|k_1) \end{pmatrix} = \\ &= \begin{pmatrix} \bar{0} \\ P_{aa}(k_1|k_1)^{-1} \cdot P_{ab}(k_1|k_1) \end{pmatrix} = \begin{pmatrix} \bar{0} \\ \mu_a \end{pmatrix} \end{aligned} \quad (2.25)$$

$$\begin{aligned} \Delta P_{AA} = P_y(k_1|k_1) - P_y(k_2|k_2) &= \begin{pmatrix} m \cdot \bar{I} & -P_{a_2a}(k_2|k_2) \\ -P_{aa_2}(k_2|k_2) & P_{aa}(k_1|k_1) - P_{aa}(k_2|k_2) \end{pmatrix} \\ \Delta X_A = y(k_2|k_2) - y(k_1|k_1) &= \begin{pmatrix} \Delta y_2 \\ \Delta y_1 \end{pmatrix} = \begin{pmatrix} \Delta X_{a_2} \\ \Delta X_a \end{pmatrix} \end{aligned} \quad (2.26)$$

Based on these analytical simplifications P_{Ab} , P_{bb} and ΔX_b are obtained

$$\begin{aligned} \mu_a &= P_{aa}(k_1|k_1)^{-1} \cdot P_{ab}(k_1|k_1) \\ \Delta P_{Ab} = \Delta P_{AA} \cdot \mu &= \begin{pmatrix} -P_{a_2a}(k_2|k_2) \cdot \mu_a \\ (P_{aa}(k_1|k_1) - P_{aa}(k_2|k_2)) \cdot \mu_a \end{pmatrix} = \begin{pmatrix} \Delta P_{a_2b} \\ \Delta P_{ab} \end{pmatrix} \\ \Delta P_{bb} = \mu^T \cdot \Delta P_{AA} \cdot \mu &= \begin{pmatrix} \bar{0} & \mu_a^T \end{pmatrix} \cdot \Delta P_{Ab} = \mu_a^T \cdot (P_{aa}(k_1|k_1) - P_{aa}(k_2|k_2)) \cdot \mu_a \\ \Delta X_b = \mu^T \cdot \Delta X_A &= \begin{pmatrix} \bar{0} & \mu_a^T \end{pmatrix} \cdot \begin{pmatrix} \Delta X_{a_2} \\ \Delta X_a \end{pmatrix} = \mu_a^T \cdot \Delta X_a \end{aligned} \quad (2.27)$$

2.3 A more special case

There is a particular case where part of the states that are directly observed during the time period $\Omega = [k_1, k_2]$ do not have dynamical evolution.

The states can be divided as

$$X = \begin{pmatrix} X_a \\ X_c \\ X_b \end{pmatrix} \quad (2.28)$$

The process model has the following structure

$$\begin{pmatrix} X_a(k+1) \\ X_c(k+1) \\ X_b(k+1) \end{pmatrix} = \begin{pmatrix} f_a(X_a(k), u(k), k) \\ X_c(k) \\ X_b(k) \end{pmatrix} \quad (2.29)$$

The observation involves part of the states. No observation involves the states X_b during the time period Ω .

$$\tilde{h}_i(X) = h_i(X_a, X_c) = \bar{0} \quad (2.30)$$

From (2.29) it is concluded that only part of the active states has to be duplicated. These states are the ones that have dynamical evolution, X_a . The active states X_c do not need to be duplicated. The augmented full system is then

$$X = \begin{pmatrix} X_{a_2} \\ X_{a_1} \\ X_c \\ X_b \end{pmatrix} \quad (2.31)$$

Where $X_{a_2}(k)$ represents the states $X_a(k)$ and the state set $X_{a_1}(k)$ represents $X_a(k_1)$ (X_a frozen at time k_1).

The reduced system will have the following state vector and associated process model

$$\begin{aligned}
 X_A = y &= \begin{pmatrix} X_{a_2} \\ X_{a_1} \\ X_c \end{pmatrix} \\
 X_{a_1}(k+1) &= X_{a_1}(k) = X_a(k_1) \\
 X_{a_2}(k+1) &= f_a(X_{a_2}(k), u(k)) \\
 X_c(k+1) &= X_c(k) = X_c(k_1)
 \end{aligned} \tag{2.32}$$

$$X_c \in R^{N_c}, \quad X_{a_1} \in R^{N_a}, \quad X_{a_2} \in R^{N_a}, \quad X_A \in R^{N_A}, \quad N_A = N_c + 2 \cdot N_a$$

The initial covariance matrix for the reduced system, at time k_1 , is based on the covariance of the 'cloned' states of the full system at the same time

$$\begin{aligned}
\text{covariance}(y(k_1|k_1), y(k_1|k_1)) &= P_y(k_1|k_1) = \begin{pmatrix} m \cdot \bar{I} & \bar{0} \\ \bar{0} & P_{[a_1,c],[a_1,c]}(k_1|k_1) \end{pmatrix} \\
P_{[a_1,c],[a_1,c]}(k_1|k_1) &= \text{covariance}\left(\begin{pmatrix} X_{a_1}(k_1|k_1) \\ X_c(k_1|k_1) \end{pmatrix}, \begin{pmatrix} X_{a_1}(k_1|k_1) \\ X_c(k_1|k_1) \end{pmatrix}\right) \\
P_{[a,c],[a,c]}(k_1|k_1) &= \text{covariance}\left(\begin{pmatrix} X_a(k_1|k_1) \\ X_c(k_1|k_1) \end{pmatrix}, \begin{pmatrix} X_a(k_1|k_1) \\ X_c(k_1|k_1) \end{pmatrix}\right) \\
P_{[a_1,c],a_2}(k_1|k_1) &= \text{covariance}\left(\begin{pmatrix} X_{a_1}(k_1|k_1) \\ X_c(k_1|k_1) \end{pmatrix}, X_{a_2}(k_1|k_1)\right) \\
P_{a_2,a_2}(k_1|k_1) &= \text{covariance}(X_{a_2}(k_1|k_1), X_{a_2}(k_1|k_1))
\end{aligned} \tag{2.33}$$

$$\begin{aligned}
P_{[a_1,c],[a_1,c]}(k_1|k_1) &= P_{[a,c],[a,c]}(k_1|k_1) \\
P_{[a_1,c],a_2}(k_1|k_1) &= \bar{0} \\
P_{a_2,a_2}(k_1|k_1) &= m \cdot \bar{I} \\
m &\longrightarrow \infty
\end{aligned}$$

It means that complete uncertainty is assumed with respect to $X_{a_2}(k_1|k_1)$.

The couple of invariant active state sets X_{a_1} and X_c are called X_α (to simplify future equations)

$$X_\alpha = \begin{pmatrix} X_{a_1} \\ X_c \end{pmatrix} \tag{2.34}$$

This set of states represent the states that have no change (according to the reduced system process model).

At the beginning of the time period $[k_1, k_2]$, time k_1 , a virtual observation is done on the reduced system,

$$X_{a_2}(k_1) = X_{a_1}(k_1) \quad (2.35)$$

This observation forces the reduced system to assume that the state sets $X_{a_2}(k_1|k_1)$ and $X_{a_1}(k_1|k_1)$ are identical at the beginning of the time period Ω .

The effect of this *v.o.* is:

$$P_y(k_1|k_1) = \begin{pmatrix} P_{a,a}(k_1|k_1) & P_{a,a}(k_1|k_1) & P_{a,c}(k_1|k_1) \\ P_{a,a}(k_1|k_1) & P_{a,a}(k_1|k_1) & P_{a,c}(k_1|k_1) \\ P_{c,a}(k_1|k_1) & P_{c,a}(k_1|k_1) & P_{c,c}(k_1|k_1) \end{pmatrix} \quad (2.36)$$

$$X_{a_2}(k_1|k_1) = X_{a_1}(k_1|k_1) = X_a(k_1|k_1)$$

where the items $X_a(k_1|k_1)$, $P_{a,a}(k_1|k_1)$ and $P_{a,c}(k_1|k_1)$ come from the full system. Then (2.36) is used as initialization of the reduced system in place of using (2.33).

After it, during the time interval Ω , all the prediction and observation steps are done over the reduced system. These prediction and observation steps are based on (2.30) and (2.29).

At the end of the time interval, time k_2 , a virtual observation (*v.o.*) is synthesized to update the full system.

The information gathered by the reduced system can be transferred to the full system by using this virtual observation.

$$h_{virtual}(X_A(k_2)) = X_A(k_2) - \xi(y(k_2)) = \bar{0} \quad (2.37)$$

$$R_{virtual} = R(P_{y,y}(k_2))$$

The objective of this *v.o.* is to produce

$$X_A(k_2|k_2) = y(k_2|k_2) \quad (2.38)$$

$$P_{A,A}(k_2|k_2) = P_{y,y}(k_2|k_2)$$

The *v.o.* involves the value of ξ and a covariance $R_{virtual}$ that do not need to be explicitly known.

The *v.o.* has a jacobian matrix

$$H = \partial h_{virtual} / \partial X = \begin{pmatrix} \partial h_{virtual} / \partial X_A & \partial h_{virtual} / \partial X_b \end{pmatrix} = \begin{pmatrix} I & \bar{0} \end{pmatrix} \in R^{N_A \cdot N} \quad (2.39)$$

The matrix update is in according to the standard KF update

$$\begin{aligned} \Delta P &= W \cdot S^{-1} \cdot W^T \\ W &= P \cdot H^T \end{aligned} \quad (2.40)$$

then

$$W = P \cdot H^T = \begin{pmatrix} P_A(k_1) & P_b(k_1) \end{pmatrix} \cdot \begin{pmatrix} \bar{I} \\ \bar{0} \end{pmatrix} = P_A(k_1) = \begin{pmatrix} P_{A,A}(k_1) \\ P_{b,A}(k_1) \end{pmatrix} = \begin{pmatrix} W_A \\ W_b \end{pmatrix}$$

$$\begin{aligned} P &= \text{covariance}(X, X) \\ P_A &= \text{covariance}(X, X_A) \\ P_b &= \text{covariance}(X, X_b) \\ P_{A,A} &= \text{covariance}(X_A, X_A) \\ P_{b,A} &= \text{covariance}(X_b, X_A) \end{aligned} \quad (2.41)$$

$$\begin{aligned}
\Delta P_{A,A} &= W_A \cdot S^{-1} \cdot W_A^T \\
W_A &= P_{A,A}(k_1) = P_{A,A}(k_1)^T = W_A^T \\
&\Downarrow \\
S^{-1} &= W_A^{-1} \cdot \Delta P_{A,A} \cdot W_A^{T^{-1}} = W_A^{-1} \cdot \Delta P_{A,A} \cdot W_A^{-1}
\end{aligned} \tag{2.42}$$

$$\begin{aligned}
\Delta P_{b,b} &= W_b \cdot S^{-1} \cdot W_b^T \\
&\Downarrow \\
\Delta P_{b,b} &= \mu \cdot \Delta P_{A,A} \cdot \mu^T \\
\mu \triangleq W_b \cdot W_A^{-1} &= P_{b,A}(k_1) \cdot P_{A,A}^{-1}(k_1)
\end{aligned} \tag{2.43}$$

$$\begin{aligned}
\Delta P_{a_2,b} &= W_{a_2} \cdot S^{-1} \cdot W_b^T \\
&\Downarrow \\
\Delta P_{a_2,b} &= P_{a_2,A}(k_1) \cdot W_A^{-1} \cdot \Delta P_{A,A} \cdot W_A^{-1} \cdot W_b^T = \\
&= P_{a_2,A}(k_1) \cdot P_{A,A}^{-1}(k_1) \cdot \Delta P_{A,A} \cdot \mu^T
\end{aligned} \tag{2.44}$$

$$\begin{aligned}
\Delta P_{c,b} &= W_c \cdot S^{-1} \cdot W_b^T \\
&\Downarrow \\
\Delta P_{c,b} &= P_{c,A}(k_1) \cdot W_A^{-1} \cdot \Delta P_{A,A} \cdot W_A^{-1} \cdot W_b^T = \\
&= P_{c,A}(k_1) \cdot P_{A,A}^{-1}(k_1) \cdot \Delta P_{A,A} \cdot \mu^T
\end{aligned} \tag{2.45}$$

considering

$$P_{A,A}(k_1) = \begin{pmatrix} \bar{I} \cdot m & \bar{0}_\alpha \\ \bar{0}_\alpha^T & P_{\alpha,\alpha}(k_1) \end{pmatrix}$$

$$P_{A,A}^{-1}(k_1) = \begin{pmatrix} \bar{I}/m & \bar{0}_\alpha \\ \bar{0}_\alpha^T & P_{\alpha,\alpha}^{-1}(k_1) \end{pmatrix}$$

$$m \longrightarrow \infty$$

(2.46)

$$\bar{0}_\alpha = \begin{pmatrix} \bar{0}_a & \bar{0}_c \end{pmatrix}$$

$$\bar{0}_c = \bar{0} \in R^{N_{a_2} \cdot N_c}, \quad \bar{0}_a = \bar{0} \in R^{N_{a_2} \cdot N_a}$$

$$\Delta P_{A,A} = P_{A,A}(k_1) - P_{A,A}(k_2) = \begin{pmatrix} -\bar{I} \cdot m & -P_{a_2,\alpha}(k_2) \\ -P_{\alpha,a_2}(k_2) & [P_{\alpha,\alpha}(k_1) - P_{\alpha,\alpha}(k_2)] \end{pmatrix}$$

then

$$\begin{aligned}
\mu &= W_b \cdot W_A^{-1} = P_{b,A}(k_1) \cdot P_{A,A}^{-1}(k_1) \\
P_{b,A}(k_1) &= \begin{pmatrix} \bar{0} & P_{b,\alpha}(k_1) \end{pmatrix} \\
\mu &= \begin{pmatrix} \bar{0} & \mu_\alpha \end{pmatrix} \\
\mu_\alpha &= P_{b,\alpha}(k_1) \cdot P_{\alpha,\alpha}(k_1)^{-1}
\end{aligned}$$

$$\begin{aligned}
P_{a_2,A}(k_1) &= \begin{pmatrix} m \cdot \bar{I} & \bar{0}_\alpha \end{pmatrix} \in R^{N_{a_2} \cdot N_A} \\
P_{a_2,A}(k_1) \cdot P_{A,A}^{-1}(k_1) &= \begin{pmatrix} \bar{I} & \bar{0}_\alpha \end{pmatrix}
\end{aligned} \tag{2.47}$$

$$\begin{aligned}
P_{c,A}(k_1) &= \begin{pmatrix} \bar{0} & P_{c,\alpha}(k_1) \end{pmatrix} \\
P_{c,A}(k_1) \cdot P_{A,A}^{-1}(k_1) &= \begin{pmatrix} \bar{0} & [P_{c,\alpha}(k_1) \cdot P_{\alpha,\alpha}^{-1}(k_1)] \end{pmatrix}
\end{aligned}$$

then

$$\begin{aligned}
\Delta P_{b,b} &= \begin{pmatrix} \bar{0} & \mu_\alpha \end{pmatrix} \cdot \Delta P_{A,A} \cdot \begin{pmatrix} \bar{0} \\ \mu_\alpha^T \end{pmatrix} \\
&= \mu_\alpha \cdot \Delta P_{\alpha,\alpha} \cdot \mu_\alpha^T
\end{aligned} \tag{2.48}$$

$$\begin{aligned}
\Delta P_{c,b} &= P_{c,A}(k_1) \cdot P_{A,A}^{-1}(k_1) \cdot \Delta P_{A,A} \cdot \mu^T = \begin{pmatrix} \bar{0} & [P_{c,\alpha}(k_1) \cdot P_{\alpha,\alpha}^{-1}(k_1)] \end{pmatrix} \cdot \Delta P_{A,A} \cdot \mu^T \\
&\quad \Downarrow \\
\Delta P_{c,b} &= P_{c,\alpha}(k_1) \cdot P_{\alpha,\alpha}^{-1}(k_1) \cdot \Delta P_{\alpha,\alpha} \cdot \mu_\alpha^T
\end{aligned} \tag{2.49}$$

$$\begin{aligned}
\Delta P_{a_2,b} &= P_{a_2,A}(k_1) \cdot P_{A,A}^{-1}(k_1) \cdot \Delta P_{A,A} \cdot \mu^T \\
P_{a_2,A}(k_1) \cdot P_{A,A}^{-1}(k_1) &= \begin{pmatrix} \bar{I} & \bar{0}_\alpha \end{pmatrix} \\
P_{a_2,A}(k_1) \cdot P_{A,A}^{-1}(k_1) \cdot \Delta P_{A,A} &= \begin{pmatrix} m \cdot \bar{I} & -P_{a_2,\alpha}(k_2) \end{pmatrix} \\
P_{a_2,A}(k_1) \cdot P_{A,A}^{-1}(k_1) \cdot \Delta P_{A,A} \cdot \mu^T &= m \cdot \bar{I} \cdot \bar{0} - P_{a_2,\alpha}(k_2) \cdot \mu_\alpha^T \\
&\Downarrow \\
\Delta P_{a_2,b} &= -P_{a_2,\alpha}(k_2) \cdot \mu_\alpha^T
\end{aligned} \tag{2.50}$$

$$\begin{aligned}
\Delta X &= K \cdot \xi \\
K &= W \cdot S^{-1} = \begin{pmatrix} K_A \\ K_b \end{pmatrix} = \begin{pmatrix} W_A \\ W_b \end{pmatrix} \cdot S^{-1} \\
\Delta X &= \begin{pmatrix} \Delta X_A \\ \Delta X_b \end{pmatrix} = \begin{pmatrix} K_A \cdot \xi \\ K_b \cdot \xi \end{pmatrix} \\
\Delta X_b &= K_b \cdot \xi = W_b \cdot S^{-1} \cdot \xi = W_b \cdot W_A^{-1} \cdot W_A \cdot S^{-1} \cdot \xi = W_b \cdot W_A^{-1} \cdot \Delta X_A \\
&\Downarrow \\
\Delta X_b &= \mu \cdot \Delta X_A = \begin{pmatrix} \bar{0} & \mu_\alpha \end{pmatrix} \cdot \begin{pmatrix} \Delta X_{a_2} \\ \Delta X_\alpha \end{pmatrix} \\
\Delta X_b &= \mu_\alpha \cdot \Delta X_\alpha
\end{aligned} \tag{2.51}$$

For the SLAM case the compressed operation is described as follows:

The state set X_a correspond to the vehicle states (they have dynamical evolution). The set X_c represents the states of the close (to the vehicle) landmarks. For a period of time the observation will involve only landmarks having position in a local region. The set X_b define the landmarks that do not belong to the local region. The population of X_c will increase when new landmarks are appended to the map provided that they are observed for first time from the vehicle.

At the beginning of the time period the vehicle states are duplicated and one of these sets is held frozen, representing the vehicle states at time k_1 . The other set represents the current vehicle states, at time k ($k_1 < k < k_2$). It has to be noticed that the transference of information from the active states to the passive states (X_b) is thanks to the past correlation between the passive states and the invariant active states, $X_{a_1}(k_1|k_1)$ and $x_c(k_1|k_1)$.

Then, at time k_1 :

consider the reduced system

$$X_A = y = \begin{pmatrix} X_{a_2} \\ X_{a_1} \\ X_c \end{pmatrix}$$

$$X_{a_1}(k+1) = X_{a_1}(k) = X_a(k_1), \quad (\text{it is 'frozen'})$$

$$X_{a_2}(k+1) = f_a(X_{a_2}(k), u(k))$$

$$X_c(k+1) = X_c(k) = X_c(k_1) \tag{2.52}$$

initial values:

$$X_{a_1}(k_1|k_1) = X_a(k_1|k_1)$$

$$X_{a_2}(k_1|k_1) = X_a(k_1|k_1)$$

$$X_c(k_1|k_1) = X_c(k_1|k_1)$$

The initial covariance matrix of the reduced system is based on the covariance of the 'cloned' states of the full system at time k_1

$$P_y(k_1|k_1) = \begin{pmatrix} P_{a,a}(k_1|k_1) & P_{a,a}(k_1|k_1) & P_{a,c}(k_1|k_1) \\ P_{a,a}(k_1|k_1) & P_{a,a}(k_1|k_1) & P_{a,c}(k_1|k_1) \\ P_{c,a}(k_1|k_1) & P_{c,a}(k_1|k_1) & P_{c,c}(k_1|k_1) \end{pmatrix} \quad (2.53)$$

During the time period $\Omega, k_1 \leq k \leq k_2$ all the prediction and observation steps are done over the reduced system considering the process model as:

$$\begin{aligned} X_{a_2}(k+1|k) &= f_a(X_{a_2}(k|k), u(k)) \\ X_{a_1}(k+1|k) &= X_{a_1}(k|k) \\ X_c(k+1|k) &= X_c(k|k) \end{aligned} \quad (2.54)$$

Any observation as a function of the vehicle states X_{a_2} and the observed landmark states:

$$h_i(X_{a_2}, X_c) = \bar{0} \quad (2.55)$$

At the end of the time period (at k_2), the global update has to be done. It is based on a virtual update and is done by executing the following steps:

(The set of states X_α represents the invariant active states, X_{a_1} and X_c .)

$$\Delta P_{\alpha,\alpha} = P_{\alpha,\alpha}(k_1|k_1) - P_{\alpha,\alpha}(k_2|k_2)$$

$$\mu_\alpha = P_{b,\alpha}(k_1|k_1) \cdot P_{\alpha,\alpha}(k_1|k_1)^{-1}$$

$$\downarrow$$

$$\Delta P_{b,b} = \mu_\alpha \cdot \Delta P_{\alpha,\alpha} \cdot \mu_\alpha^T$$

$$\Delta P_{c,b} = P_{c,\alpha}(k_1|k_1) \cdot P_{\alpha,\alpha}^{-1}(k_1|k_1) \cdot \Delta P_{\alpha,\alpha} \cdot \mu_\alpha^T \quad (2.56)$$

$$\Delta P_{a,b} = -P_{a_2,\alpha}(k_2|k_2) \cdot \mu_\alpha^T$$

$$P_{a,c}(k_2|k_2) = P_{a_2,c}(k_2|k_2)$$

$$\Delta X_b = \mu_\alpha \cdot \Delta X_\alpha$$

$$X_a(k_2|k_2) = X_{a_2}(k_2|k_2)$$

$$X_c(k_2|k_2) = X_c(k_2|k_2)$$

Please notice that $P_{a,b}$, $P_{b,c}$, $P_{a,c}$, X_a , ΔX_b are related to the full system and $P_{b,\alpha}$, $P_{\alpha,\alpha}$, $P_{a_2,\alpha}$, $P_{a_2,c}$, X_{a_2} , ΔX_α are related to the reduced system.

Additionally it is important to remark that the internal estimator, running over the reduced system, can be other filter than an EKF. In that case $P_{\alpha,\alpha}(k_2, k_2)$, $P_{a_2,\alpha}(k_2|k_2)$, $X_\alpha(k_2|k_2)$ have to be obtained by doing a gaussian approximation of the reduced system probability distribution at time k_2 and having guaranteed the positiveness of $\Delta P_{\alpha,\alpha}$ (the uncertainty of the invariant active states can not increase, $P_{\alpha,\alpha}(k_1|k_1) \geq P_{\alpha,\alpha}(k_2|k_2)$). A standard EKF will always meet this condition, but a too conservative gaussian approximation of a

probability distribution can generate the mentioned inconsistency.

2.4 Advantages of the hybrid compressed filters

As was presented before a compressed filter implementation will involve 2 estimators. The internal estimator will run over a reduced system. Due to the low order number of estimated states of the reduced system this filter will be able to run at high frequency. Additionally adequate non linear and numerically stable techniques can be considered to be used in the internal estimator. Those techniques are usually too expensive to be applied to the full estimation problem but they are feasible of being applied in real time over a low order system.

The external estimator is always an EKF estimator.

For the internal loop a more sophisticated estimator can be used. This will have the following advantages:

1. High frequency operation.
2. Improved management of 'short term' non linear effects.
3. Improved numerical stability.

High frequency operation is possible provided that the reduced system dimension is small. Non linear effects of type 'short term' are those that are present for short periods of time. For instance landmark states initialization in bearing-only SLAM or in range-only SLAM. A pure EKF (or another estimator assuming gaussian probability distributions) can not represent these initial landmark probability distribution based on a set of observations from the same or very similar viewpoints. An estimator that does not assume gaussian p.d. can perform the proper initial representation and after several observations from different viewpoints be able to have a good gaussian approximation of the p.d. and transfer this

p.d. to the external EKF filter. A typical non desirable non linear behavior in the SLAM estimation process is due to high deviation on the estimated vehicle heading. If this large error occurs during a short period of time then the internal filter will be able to manage it. If the heading high deviation persists for a long time period the external filter will have to deal with this high deviation. Since the external filter is an EKF estimator it will not be able to satisfactorily operate under this condition.

Numerical stability can be remarkably improved. The internal estimation loop can be based in numerically stable techniques than can be too expensive to be directly applied to the full system. Some square root KF versions have excellent numerical stability characteristics (in comparison to the standard EKF). Their application in a reduced estimation loop is real time feasible.

2.5 Summary

This chapter has presented a generalization of the Compressed Filter idea introduced in the previous chapter.

The compressed filter can remarkably decrease the computational complexity of the estimation problem. Additionally it enables the hybrid combination of an EKF estimator, that is applied to the full system, with other non linear estimators that operate locally on small sets of states and at high frequency. The capacity to manage high frequency sequences of predictions and updates represents a remarkable improvement respect to the standard full filters. Important improvements in the numerical stability and non linear treatment can be achieved provided that sophisticated estimators can be applied on the internal estimation loop due to its reduced order.